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INDEPENDENT AVERAGE CONNECTIVITY NUMBER IN GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a simple, undirected graph. The vulnerability of a graph is a determination that includes certain properties of the graph not to be damaged after the removal of a number of vertices or edges. In this paper, the concept of independent average connectivity number is introduced as a new vulnerability measure and some bounds and independent average connectivity number of some well-known graph families are examined. Independent average connectivity will give the number of vertices that need to be removed to separate the pairs of vertices selected from the independent set in a graph $G(V, E)$. Independent average connectivity number of a graph $G(V, E)$ is denoted by $\overline{k}_\beta(G)$ and defined as $\overline{k}_\beta(G) = \frac{TI(G)}{\binom{n}{2}}$ where $TI(G)$ is a total independence number of $G(V, E)$ graph with n vertices. In this study, all the graphs considered simple, finite, and undirected.

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1 Introduction

The stability and reliability of a network are of prime importance to network designers. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown. As the network begins losing connection lines or centers, eventually, there is a loss of effectiveness. If the communication network is modelled as a simple, undirected, connected, and unweighted graph G deterministic measures tend to provide the worst-case analysis of some aspects of the overall disconnection process (Barefoot et al., 1994; Cozzens, 1987).

If we consider the graph G as a model of a network, various measurements are needed to assess the vulnerability of this graph. These include connectivity (Mader, 1979), edge-connectivity (Oellermann, 1996), and average connectivity (Henning & Oellermann, 2004). And recently, several interesting works on vulnerability parameters such as average lower reinforcement number (Turacı & Aslan, 2016), average covering number (Dogan & Dundar, 2013) have been defined. In these parameters, connectivity is a well-known concept in network design and it has been used in many applications and in many areas of science. Average connectivity has also proven to be a useful measure for real-world networks, including street networks (Boeing, 2017) and communication networks (Rak et al., 2015). In this paper, independent average connectivity was considered.

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The connectivity of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph and is denoted by $k(G)$ (Mader, 1979). The edge connectivity number $k'(G)$ of a graph G is defined as the minimum number of edges that need to be removed from the graph to make it disconnected (Oellermann, 1996).

These parameters have been extensively studied. Recent interest in the vulnerability and reliability of networks has given rise to a host of other measures, some of which are more global in nature. Beineke et. al, introduced a parameter to give a more refined measure of the global "amount" of connectivity in 2002 (Beineke et al., 2002). The average connectivity of a graph G with n vertices, denoted by $\overline{k}(G)$, is defined as $\overline{k}(G) = \frac{\sum_{\{u,v\}} k_G(u,v)}{\binom{n}{2}} = \frac{k(G)}{\binom{n}{2}}$ where $k_G(u,v)$ is the minimum number of vertices whose deletion makes v unreachable from u. (The expression $\sum_{\{u,v\}} k_G(u,v)$ is sometimes referred to as the total connectivity of G) (Beineke et al., 2002; Dankelmann & Oellermann, 2003; Henning & Oellermann, 2001). By Menger's theorem, $k(u,v)$ is equal to the maximum number of internally disjoint paths joining u and v (Chartrand & Lesniak, 2005). Note that the relationship between connectivity and average connectivity is $\overline{k}(G) \geq k(G)$ (Beineke et al., 2002). Average connectivity is much more attractive for applications because it can be computed in polynomial time whereas other global parameters, such as toughness and integrity, are NP-hard computationally. There are more global parameters to investigate vulnerability. For another example of global parameters, see (Blidia et al., 2005; Chung, 1988).

Throughout this paper, unless otherwise specified, a graph G is denoted by $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are vertex and edge sets of G, respectively. In the graph G, n and m denote the number of vertices and the number of edges, respectively. G is a simple, connected, and undirected graph of order $n \geq 2$, and also we use the terminology of (Chartrand & Lesniak, 2005).

Proposition 1. *The connectivity of a graph does not exceed its minimum vertex degree.* $k(G) \leq k'(G) \leq \delta(G)$ *(Buckley & Harary, 1990).*

Proposition 2. *In a graph G, if there are at least k internally disjoint paths between every pair of vertices u and v, then G is said to be k-connected (Buckley & Harary, 1990).*

2 Bounds for the Independent Average Connectivity Number

In this section, we give a definition of the independent average connectivity number and some bounds for independent average connectivity number.

Definition 1. *Two vertices that are not adjacent in a graph G are said to be independent. A set S of vertices is independent if every two vertices of S are independent. The vertex independent number or simply the independence number $\beta(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G (Chartrand & Lesniak, 2005).*

Definition 2. *Let $S_1, S_2, \ldots, S_r$ be sets that represent the independence number of a graph G. The Total Independence ($TI(G)$) of the graph, based on these sets S_i ($i = (1, r)$), is defined as,*

$$TI(G) = \min_i \left\{ \sum_{(u,v)\in S_i} k_G(u,v) \right\}.$$

(Gürtunca, 2007).

Definition 3. *Independent Average Connectivity is defined as,*

$$\overline{k}_\beta(G) = \frac{TI(G)}{\binom{n}{2}},$$

where n is the number of vertices (Gürtunca, 2007).

If two graphs have the same connectivity number and average connectivity number, then these parameters are not enough to distinguish them. In this case, a new parameter which will distinguish them is needed. Can independent average connectivity number be regarded as a vulnerability parameter that compares these graphs? Let's see this with a simple comparison between the graphs G and H.

Let the graphs G and H with 5 vertices as shown in Figure 1 have $k(G) = 1$ and $k(H) = 1$. It can easily be checked that graphs G and H have not only equal connectivity number but also equal average connective numbers which are $\overline{k}(G) = \frac{1}{10}$ and $\overline{k}(H) = \frac{1}{10}$.

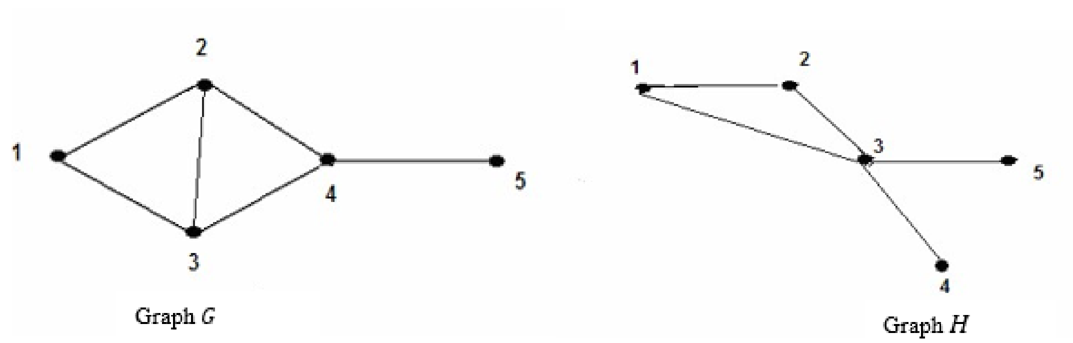

Figure 1: Graphs G and H with 5 vertices

It follows that $\overline{k}_\beta(G) = \frac{\min_i\{2,1,1,1\}}{\binom{5}{2}} = \frac{1}{10}$. The reader can check that $\overline{k}_\beta(H) = \frac{3}{10}$.

As a result, according to the independent average connectivity, graph G is more stable than graph H.

Theorem 1. *The Total Independence number of a connected n-vertex graph G is*

$$TI(G) = \binom{\beta(G)}{2} \cdot \delta(G).$$

Proof. According to the definition of Connectivity, $k(G) = \min\{k_G(u,v) : u, v \in V\}$.

In other words, for each pair (u,v) in $TI(G)$, $TI(G) = \min_i \left\{\sum_{(u,v)\in S_i} k_G(u,v)\right\}$, $TI(G) = \binom{\beta(G)}{2} \cdot k(G)$.

If we substitute $k(G)$ for $k_G(u,v)$ in the expression above, a graph's connectivity cannot exceed its minimum vertex degree; hence, from Proposition 1 $k(G) \le k'(G) \le \delta(G)$. Therefore,

$$TI(G) = \binom{\beta(G)}{2} \cdot \delta(G).$$

□

Theorem 2. *For a connected n-vertex graph G,*

$$\overline{k}(G) \ge \overline{k}_\beta(G).$$

Proof. $\overline{k}(G)$ is defined over all vertex pairs of graph G (Dankelmann & Oellermann, 2003). When $\beta(G) < n$, and from definitions 3 and 6, $\overline{k}(G) \ge \overline{k}_\beta(G)$ is obtained. □

Theorem 3. *For any connected graph G,*

$$k(G) > \overline{k}_\beta(G).$$

Proof. In definition 1, we consider all vertex pairs in determining the connectivity of a graph G. In definition 7, $\overline{k}_\beta(G)$ calculates the number of internally disjoint paths between all pairs of vertices in the independent set of G and divides their sum $\binom{n}{2}$.

As a result, $k(G) > \overline{k}_\beta(G)$. □

Theorem 4. *For any connected graph G, $\overline{k}(G) \geq k(G) > \overline{k}_\beta(G)$.*

Proof. From Theorem 2, we have $\overline{k}(G) \geq \overline{k}_\beta(G)$. From Theorem 3, we have $k(G) > \overline{k}_\beta(G)$. Clearly, for any graph G, $\overline{k}(G) \geq k(G)$ (Beineke et al., 2002).

Therefore, $\overline{k}(G) \geq k(G) > \overline{k}_\beta(G)$. □

3 Independent Average Connectivity Number For Common Graph Families

In this section, independent average connectivity numbers of well-known classes of graphs are determined.

Theorem 5. *The independent average connectivity number of a path is,*

$$\overline{k}_\beta(P_n) = \frac{TI(P_n)}{\binom{n}{2}} = \frac{\binom{\beta(P_n)}{2}}{\binom{n}{2}}.$$

Proof. P_n is a path graph with n vertices. Let's determine the elements of set S that give the independence number of this graph. The number of elements in the set giving the independence number of P_n graph is:

$$\beta(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \text{ is odd,} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

To make P_n disconnected or to obtain a single isolated vertex from the graph, the minimum number of vertices to be removed is selected from the elements of set S. The set that gives the independence number of P_n graph can exist in multiple sets of S. Therefore, the total number of internally disjoint paths between all pairs of vertices in the sets giving the independence number of P_n is:

$$TI(P_n) = \min_i \left\{ \sum_{(u,v)\in S_i} k_{P_n}(u,v) \right\} = \binom{\beta(P_n)}{2}.$$

The result obtained from dividing the total number of all pairs of vertices in the total independence set will give us the independent average connectivity number of P_n. Therefore, the independent average connectivity number of P_n is:

$$\overline{k}_\beta(P_n) = \frac{TI(P_n)}{\binom{n}{2}} = \frac{\binom{\beta(P_n)}{2} \cdot 2}{\binom{n}{2}}.$$

□

Theorem 6. *The independent average connectivity number of a cycle is,*

$$\overline{k}_\beta(C_n) = \frac{TI(C_n)}{\binom{n}{2}} = \frac{\binom{\beta(C_n)}{2} \cdot 2}{\binom{n}{2}}.$$

Proof. C_n is a complete graph with n vertices. Let's determine the elements of set S that give the independence number of this graph. The number of elements in the set giving the independence number of C_n graph is:

$$\beta(C_n) = \left\lfloor \frac{n}{2} \right\rfloor .$$

To make C_n disconnected, the minimum number of vertices that need to be removed from the graph is always 2, as the graph contains a cycle. For these pairs of vertices selected from the independent set,

$$TI(C_n) = \min\left\{\sum k_\beta C_n(u,v)\right\} = \binom{\beta(C_n)}{2} \cdot 2.$$

So the independent average connectivity number of C_n is:

$$\overline{k}_\beta(C_n) = \frac{TI(C_n)}{\binom{n}{2}} = \frac{\binom{\beta(C_n)}{2} \times 2}{\binom{n}{2}}.$$

□

Theorem 7. *The independent average connectivity number of a star $K_{1,n-1}$ is,*

$$\overline{k}_\beta(K_{1,n-1}) = \frac{TI(K_{1,n-1})}{\binom{n}{2}} = \frac{\binom{\beta(K_{1,n-1})}{2}}{\binom{n}{2}}.$$

Proof. $K_{1,n-1}$ is a star with n vertices. First, determine the elements of set S that give the independence number of this graph.

The minimum number of vertices that need to be removed from the connected $K_{1,n-1}$ graph to make it disconnected is $\beta(K_{1,n-1}) = n-1$. The sum of the internally disjoint paths between pairs of vertices selected from the independent set is,

$$TI(K_{1,n-1}) = \min\left\{\sum k_\beta(K_{1,n-1})(u,v)\right\} = \binom{\beta(K_{1,n-1})}{2}.$$

So the independent average connectivity number of $K_{1,n-1}$ graph is:

$$\overline{k}_\beta(K_{1,n-1}) = \frac{TI(K_{1,n-1})}{\binom{n}{2}} = \frac{\binom{\beta(K_{1,n-1})}{2}}{\binom{n}{2}} = \frac{n-2}{n}.$$

Additionally, the maximum vertex degree of $K_{1,n-1}$ is $\Delta(K_{1,n-1}) = n-1$, and the minimum vertex degree is $\delta(K_{1,n-1}) = 1$.

The result for the independent average connectivity number of $K_{1,n-1}$ is,

$$\bar{k}_\beta(K_{1,n-1}) = \frac{\Delta(K_{1,n-1}) - \delta(K_{1,n-1})}{n} = \frac{n-1-1}{n} = \frac{n-2}{n}.$$

□

Theorem 8. *The independent average connectivity number of a bipartite graph is,*

$$\overline{k}_\beta(K_{m,n}) = \frac{TI(K_{m,n})}{\binom{m+n}{2}} = \frac{\binom{\beta(K_{m,n})}{2} \times \delta(K_{m,n})}{\binom{m+n}{2}}, \ \textit{for } m \leq n.$$

Proof. Let $K_{m,n}$ be a bipartite graph. First determine the elements of set S for this graph.

The vertex set of $K_{m,n}$, denoted by v, is divided into two subsets, where $|v_1| = m$ and $|v_2| = n$.

If $m \leq n$, then the independence number of graph $K_{m,n}$ is $\beta(K_{m,n}) = m$. The minimum number of vertices that need to be removed from the graph to make it disconnected is equal to the independence number of the bipartite graph, is $\beta(K_{m,n}) = m$.

The sum of the total connectivity of all pairs of vertices selected from the independent set, which consists of $|v_2| = m$ vertices from $\beta(K_{m,n}) = m$, is the total connectivity of the graph, denoted as

$$TI(K_{m,n}) = \min\left\{\sum k_\beta(K_{m,n}(u,v))\right\} = \left\{\binom{\beta(K_{m,n})}{2}\right\} \cdot \delta(K_{m,n}).$$

So the independent average connectivity number of $K_{m,n}$ is

$$\bar{k}_\beta(K_{m,n}) = \frac{TI(K_{m,n})}{\binom{m+n}{2}} = \frac{\left\{\binom{\beta(K_{m,n})}{2}\right\} \cdot \delta(K_{m,n})}{\left\{\binom{m+n}{2}\right\}}.$$

□

Theorem 9. *The independent average connectivity number of a wheel is*

$$\bar{k}_\beta(W_{1,n}) = \frac{TI(W_{1,n})}{\binom{n+1}{2}} = \frac{\binom{\beta(W_{1,n})}{2} \cdot 3}{\binom{n+1}{2}}.$$

Proof. $W_{1,n}$ is a wheel graph ($n \geq 3$). First, determine the elements of the independent set for this graph has one vertex with degree n and all other vertices have degree 3.

Since $W_{1,n} = K_1 + C_n$, we can take elements of the independent set from C_n. The independence number of $W_{1,n}$ can be determined by the independence number of C_n.

So the independence number of the wheel graph is $\beta(W_{1,n}) = \lceil n/2 \rceil$. The sum of 3 times the total number of pairs of vertices selected from the elements of the independence set, which gives the independence number, is the total independence, denoted as

$$TI(W_{1,n}) = \min\left\{\sum k_\beta(W_{1,n}(u,v))\right\} \cdot 3.$$

The result for the independent average connectivity number of the $W_{1,n}$ graph is

$$\bar{k}_\beta(W_{1,n}) = \frac{TI(W_{1,n})}{\binom{n+1}{2}} = \frac{\binom{\beta(W_{1,n})}{2} \cdot 3}{\binom{n+1}{2}}.$$

□

4 Conclusion

In this paper, the independent average connectivity number is introduced as a new vulnerability parameter. The independent average connectivity number of basic graph classes P_n, C_n, W_n, $K_{1,n}$, $K_{m,n}$ are examined, and some bounds are given. In future work, we plan to generalize the results of the independent average connectivity number under basic operation classes.

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